14-2 Permutations and Combinations

What You'll Learn

- Determine probabilities using permutations.
- Determine probabilities using combinations.

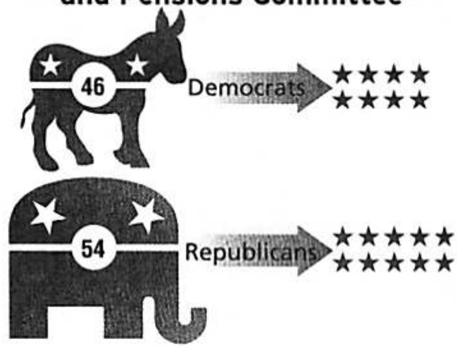
Vocabulary

- permutation
- combination

How can combinations be used to form committees?

The United States Senate forms various committees by selecting senators from both political parties. The Senate Health, Education, Labor, and Pensions Committee of the 106th Congress was made up of 10 Republican senators and 8 Democratic senators. How many different ways could the committee have been selected? The members of the committee were selected in no particular order. This is an example of a combination.

Senate Health, Education, Labor, and Pensions Committee



PERMUTATIONS An arrangement or listing in which order or placement is important is called a **permutation**.

Example 1 Tree Diagram Permutation

EMPLOYMENT The manager of a coffee shop needs to hire two employees, one to work at the counter and one to work at the drive-through window. Katie, Bob, Alicia, and Jeremiah all applied for a job. How many possible ways are there for the manager to place the applicants?

Use a tree diagram to show the possible arrangements.

Counter	Drive-Through	Outcomes
Katie (K)	Bob	KB
	Alicia	KA
	Jeremiah	KJ
Bob (B)	Katie	BK
	Alicia	BA
	Jeremiah	BJ
Alicia (A)	Jeremiah	AJ
	Katie	AK
	Bob	AB
Jeremiah (J)	Katie	JK
	Bob	JB
	Alicia	JA

There are 12 different ways for the 4 applicants to hold the 2 positions.

> **Study Tip**
>
> *Common Misconception*
> When arranging two objects *A* and *B* using a permutation, the arrangement *AB* is different from the arrangement *BA*.

In Example 1, the positions are in a specific order, so each arrangement is unique. The symbol ${}_4P_2$ denotes the number of permutations when arranging 4 applicants in 2 positions. You can also use the Fundamental Counting Principle to determine the number of permutations.

$$
\begin{aligned}
{}_4P_2 &= \underbrace{4}_{\text{ways to choose first employee}} \cdot \underbrace{3}_{\text{ways to choose second employee}} \\
&= 4 \cdot 3 \cdot \frac{2 \cdot 1}{2 \cdot 1} && \frac{2 \cdot 1}{2 \cdot 1} = 1 \\
&= \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1} && \text{Multiply.} \\
&= \frac{4!}{2!} && 4 \cdot 3 \cdot 2 \cdot 1 = 4!,\ 2 \cdot 1 = 2!
\end{aligned}
$$

In general, ${}_nP_r$ is used to denote the number of permutations of n objects taken r at a time.

Key Concept — *Permutation*

- **Words** The number of permutations of n objects taken r at a time is the quotient of $n!$ and $(n - r)!$.
- **Symbols** ${}_nP_r = \frac{n!}{(n-r)!}$

Example 2 Permutation

Find ${}_{10}P_6$.

${}_nP_r = \frac{n!}{(n-r)!}$	Definition of ${}_nP_r$
${}_{10}P_6 = \frac{10!}{(10-6)!}$	$n = 10, r = 6$
${}_{10}P_6 = \frac{10!}{4!}$	Subtract.
${}_{10}P_6 = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot \overset{1}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}}{\underset{1}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}}$	Definition of factorial
${}_{10}P_6 = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ or 151,200	Simplify.

There are 151,200 permutations of 10 objects taken 6 at a time.

Permutations are often used to find the probability of events occurring.

Example 3 Permutation and Probability

A word processing program requires a user to enter a 7-digit registration code made up of the digits 1, 2, 4, 5, 6, 7, and 9. Each number has to be used, and no number can be used more than once.

a. How many different registration codes are possible?

Since the order of the numbers in the code is important, this situation is a permutation of 7 digits taken 7 at a time.

${}_nP_r = \frac{n!}{(n-r)!}$	Definition of permutation
${}_7P_7 = \frac{7!}{(7-7)!}$	$n = 7, r = 7$
${}_7P_7 = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1}$ or 5040	Definition of factorial

There are 5040 possible codes with the digits 1, 2, 4, 5, 6, 7, and 9.

Study Tip

Permutations
The number of permutations of n objects taken n at a time is $n!$.
${}_nP_n = n!$

b. What is the probability that the first three digits of the code are even numbers?

Use the Fundamental Counting Principle to determine the number of ways for the first three digits to be even.

- There are 3 even digits and 4 odd digits.
- The number of choices for the first three digits, if they are even, is $3 \cdot 2 \cdot 1$.
- The number of choices for the remaining odd digits is $4 \cdot 3 \cdot 2 \cdot 1$.
- The number of favorable outcomes is $3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ or 144. There are 144 ways for this event to occur out of the 5040 possible permutations.

$$P(\text{first 3 digits even}) = \frac{144}{5040} \quad \begin{matrix} \leftarrow \text{number of favorable outcomes} \\ \leftarrow \text{number of possible outcomes} \end{matrix}$$

$$= \frac{1}{35} \quad \text{Simplify.}$$

The probability that the first three digits of the code are even is $\frac{1}{35}$ or about 3%.

Study Tip

Look Back
To review **probability**, see Lesson 2-6.

COMBINATIONS An arrangement or listing in which order is not important is called a **combination**. For example, if you are choosing 2 salad ingredients from a list of 10, the order in which you choose the ingredients does not matter.

Key Concept — *Combination*

- **Words** The number of combinations of n objects taken r at a time is the quotient of $n!$ and $(n - r)!r!$.
- **Symbols** $_nC_r = \frac{n!}{(n - r)!r!}$

Standardized Test Practice

Example 4 *Combination*

Multiple-Choice Test Item

The students of Mr. DeLuca's homeroom had to choose 4 out of the 7 people who were nominated to serve on the Student Council. How many different groups of students could be selected?

(A) 840 (B) 210

(C) 35 (D) 24

Read the Test Item

The order in which the students are chosen does not matter, so this situation represents a combination of 7 people taken 4 at a time.

Solve the Test Item

$$_nC_r = \frac{n!}{(n - r)!r!} \quad \text{Definition of combination}$$

$$_7C_4 = \frac{7!}{(7 - 4)!4!} \quad n = 7, r = 4$$

$$= \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}^{1}}{3 \cdot 2 \cdot 1 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}_{1}} \quad \text{Definition of factorial}$$

$$= \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} \text{ or } 35 \quad \text{Simplify.}$$

There are 35 different groups of students that could be selected. Choice C is correct.

The Princeton Review

Test-Taking Tip

Read each question carefully to determine whether the situation involves a permutation or a combination. Often, the answer choices include examples of both.

Combinations and the products of combinations can be used to determine probabilities.

Example 5 Use Combinations

SCHOOL A science teacher at Sunnydale High School needs to choose 12 students out of 16 to serve as peer tutors. A group of 7 seniors, 5 juniors, and 4 sophomores have volunteered to be tutors.

a. How many different ways can the teacher choose 12 students?

The order in which the students are chosen does not matter, so we must find the number of combinations of 16 students taken 12 at a time.

$${}_nC_r = \frac{n!}{(n-r)!r!} \quad \text{Definition of combination}$$

$${}_{16}C_{12} = \frac{16!}{(16-12)!12!} \quad n = 16, r = 12$$

$$= \frac{16!}{4!12!} \quad 16 - 12 = 4$$

$$= \frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot \overset{1}{\cancel{12!}}}{4! \cdot \underset{1}{\cancel{12!}}} \quad \text{Divide by the GCF, 12!.}$$

$$= \frac{43{,}680}{24} \text{ or } 1820 \quad \text{Simplify.}$$

There are 1820 ways to choose 12 students out of 16.

b. If the students are chosen randomly, what is the probability that 4 seniors, 4 juniors, and 4 sophomores will be selected?

There are three questions to consider.

- How many ways can 4 seniors be chosen from 7?
- How many ways can 4 juniors be chosen from 5?
- How many ways can 4 sophomores be chosen from 4?

Using the Fundamental Counting Principle, the answer can be determined with the product of the three combinations.

$$\underbrace{({}_7C_4)}_{\text{ways to choose 4 seniors out of 7}} \cdot \underbrace{({}_5C_4)}_{\text{ways to choose 4 juniors out of 5}} \cdot \underbrace{({}_4C_4)}_{\text{ways to choose 4 sophomores out of 4}}$$

$$({}_7C_4)({}_5C_4)({}_4C_4) = \frac{7!}{(7-4)!4!} \cdot \frac{5!}{(5-4)!4!} \cdot \frac{4!}{(4-4)!4!} \quad \text{Definition of combination}$$

$$= \frac{7!}{3!4!} \cdot \frac{5!}{1!4!} \cdot \frac{4!}{0!4!} \quad \text{Simplify.}$$

$$= \frac{7 \cdot 6 \cdot 5}{3!} \cdot \frac{5}{1} \quad \text{Divide by the GCF, 4!.}$$

$$= 175 \quad \text{Simplify.}$$

There are 175 ways to choose this particular combination out of 1820 possible combinations.

$$P(\text{4 seniors, 4 juniors, 4 sophomores}) = \frac{175}{1820} \quad \begin{array}{l} \leftarrow \text{number of favorable outcomes} \\ \leftarrow \text{number of possible outcomes} \end{array}$$

$$= \frac{5}{52} \quad \text{Simplify.}$$

The probability that the science teacher will randomly select 4 seniors, 4 juniors, and 4 sophomores is $\frac{5}{52}$ or about 10%.

Study Tip

Combinations
The number of combinations of n objects taken n at a time is 1.
${}_nC_n = 1$

www.algebra1.com/extra_examples

Check for Understanding

Concept Check

1. **OPEN ENDED** Describe the difference between a permutation and a combination. Then give an example of each.

2. **Demonstrate** and explain why ${}_nC_r = 1$ whenever $n = r$. What does ${}_nP_r$ always equal when $n = r$?

3. **FIND THE ERROR** Eric and Alisa are taking a trip to Washington, D.C. Their tour bus stops at the Lincoln Memorial, the Jefferson Memorial, the Washington Monument, the White House, the Capitol Building, the Supreme Court, and the Pentagon. Both are finding the number of ways they can choose to visit 5 of these 7 sites.

Eric	Alisa
${}_7C_5 = \frac{7!}{2!}$ or 2520	${}_7C_5 = \frac{7!}{2!\,5!}$ or 21

Who is correct? Explain your reasoning.

Guided Practice

Determine whether each situation involves a *permutation* or *combination*. Explain your reasoning.

4. choosing 6 books from a selection of 12 for summer reading
5. choosing digits for a personal identification number

Evaluate each expression.

6. ${}_8P_5$
7. ${}_7C_5$
8. $({}_{10}P_5)({}_3P_2)$
9. $({}_6C_2)({}_4C_3)$

For Exercises 10–12, use the following information.
The digits 0 through 9 are written on index cards. Three of the cards are randomly selected to form a 3-digit code.

10. Does this situation represent a permutation or a combination? Explain.
11. How many different codes are possible?
12. What is the probability that all 3 digits will be odd?

Standardized Test Practice

13. A diner offers a choice of two side items from the list with each entrée. How many ways can two items be selected?

Ⓐ 15　　Ⓑ 28
Ⓒ 30　　Ⓓ 56

Side Items	
French fries	mixed vegetables
baked potato	rice pilaf
cole slaw	baked beans
small salad	applesauce

Practice and Apply

Determine whether each situation involves a *permutation* or *combination*. Explain your reasoning.

14. team captains for the soccer team
15. three mannequins in a display window
16. a hand of 10 cards from a selection of 52
17. the batting order of the New York Yankees

18. first place and runner-up winners for the table tennis tournament
19. a selection of 5 DVDs from a group of eight
20. selection of 2 candy bars from six equally-sized bars
21. the selection of 2 trombones, 3 clarinets, and 2 trumpets for a jazz combo

Evaluate each expression.

22. ${}_{12}P_3$	23. ${}_{4}P_1$	24. ${}_{6}C_6$
25. ${}_{7}C_3$	26. ${}_{15}C_3$	27. ${}_{20}C_8$
28. ${}_{15}P_3$	29. ${}_{16}P_5$	30. $({}_{7}P_7)({}_{7}P_1)$
31. $({}_{20}P_2)({}_{16}P_4)$	32. $({}_{3}C_2)({}_{7}C_4)$	33. $({}_{8}C_5)({}_{5}P_5)$

SOFTBALL **For Exercises 34 and 35, use the following information.**
The manager of a softball team needs to prepare a batting lineup using her nine starting players.

34. Does this situation involve a permutation or a combination?
35. How many different lineups can she make?

SCHOOL **For Exercises 36–39, use the following information.**
Mrs. Moyer's class has to choose 4 out of 12 people to serve on an activity committee.

36. Does the selection of the students involve a permutation or a combination? Explain.
37. How many different groups of students could be selected?
38. Suppose the students are selected for the positions of chairperson, activities planner, activity leader, and treasurer. How many different groups of students could be selected?
39. What is the probability that any one of the students is chosen to be the chairperson?

GAMES **For Exercises 40–42, use the following information.**
In your turn of a certain game, you roll five different-colored dice.

40. Do the outcomes of rolling the five dice represent a permutation or a combination? Explain.
41. How many outcomes are possible?
42. What is the probability that all five dice show the same number on one roll?

BUSINESS **For Exercises 43 and 44, use the following information.**
There are six positions available in the research department of a software company. Of the applicants, 15 are men and 10 are women.

43. In how many ways could 4 men and 2 women be chosen if each were equally qualified?
44. What is the probability that five women are selected if the positions are randomly filled?

TRACK **For Exercises 45 and 46, use the following information.**
Central High School is competing against West High School at a track meet. Each team entered 4 girls to run the 1600-meter event. The top three finishers are awarded medals.

45. How many different ways can the runners place first, second, and third?
46. If all eight runners have an equal chance of placing, what is the probability that the first and second place finishers are from West and the third place finisher is from Central?

Homework Help

For Exercises	See Examples
14–21, 34, 36, 40	1, 4
22–33, 35, 37–39, 41–49	2, 3, 5

Extra Practice

See page 851.

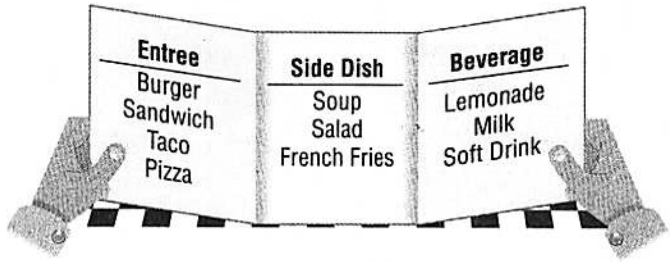
More About...

Softball

The game of softball was developed in 1888 as an indoor sport for practicing baseball during the winter months.

Source: www.encyclopedia.com

DINING **For Exercises 47–49, use the following information.**
For lunch in the school cafeteria, you can select one item from each category to get the daily combo.



47. Find the number of possible meal combinations.

48. If a side dish is chosen at random, what is the probability that a student will choose soup?

49. What is the probability that a student will randomly choose a sandwich and soup?

CRITICAL THINKING **For Exercises 50 and 51, use the following information.**
Larisa is trying to solve a word puzzle. She needs to arrange the letters H, P, S, T, A, E, and O into a two-word arrangement.

50. How many different arrangements of the letters can she make?

51. Assuming that each arrangement has an equal chance of occurring, what is the probability that she will form the words *tap shoe* on her first try?

WebQuest

You can use permutations and combinations to analyze data on U.S. schools. Visit **www.algebra1.com/webquest** to continue work on your WebQuest project.

SWIMMING **For Exercises 52–54, use the following information.**
A swimming coach plans to pick four swimmers out of a group of 6 to form the 400-meter freestyle relay team.

52. How many different teams can he form?

53. The coach must decide in which order the four swimmers should swim. He timed the swimmers in each possible order and chose the best time. How many relays did the four swimmers have to swim so that the coach could collect all of the data necessary?

54. If Tomás is chosen to be on the team, what is the probability that he will swim in the third leg?

55. WRITING IN MATH Answer the question that was posed at the beginning of the lesson.

How can combinations be used to form committees?

Include the following in your answer:

- a few sentences explaining why forming a Senate committee is a combination, and
- an explanation of how to find the number of ways to select the committee if committee positions are based upon seniority.

Standardized Test Practice

56. There are 12 songs on a CD. If 10 songs are played randomly and each song is played once, how many arrangements are there?

(A) 479,001,600 (B) 239,500,800 (C) 66 (D) 1

57. Julie remembered that the 4 digits of her locker combination were 4, 9, 15, and 22, but not their order. What is the maximum number of attempts Julie has to make to find the correct combination?

(A) 4 (B) 16 (C) 24 (D) 256

Maintain Your Skills

Mixed Review

58. The Sanchez family acts as a host family for a foreign exchange student during each school year. It is equally likely that they will host a girl or a boy. How many different ways can they host boys and girls over the next four years? *(Lesson 14-1)*

STATISTICS For Exercises 59–62, use the table at the right. *(Lesson 13-5)*

59. Make a box-and-whisker plot of the data.
60. What is the range of the data?
61. Identify the lower and upper quartiles.
62. Name any outliers.

Highest Paying Occupations in America

Occupation	Median Salary
Physician	$148,000
Dentist	$93,000
Lobbyist	$91,300
Management Consultant	$61,900
Lawyer	$60,500
Electrical Engineer	$59,100
School Principal	$57,300
Aeronautical Engineer	$56,700
Airline Pilot	$56,500
Civil Engineer	$55,800

Source: U.S. Bureau of Labor Statistics

Online Research **Data Update** For current data on the highest-paying occupations, visit: www.algebra1.com/data_update

Simplify each expression. *(Lesson 12-2)*

63. $\dfrac{x+3}{x^2+6x+9}$
64. $\dfrac{x^2-49}{x^2-2x-35}$
65. $\dfrac{n^2-n-20}{n^2+9n+20}$

Find the distance between each pair of points whose coordinates are given. Express answers in simplest radical form and as decimal approximations rounded to the nearest hundredth if necessary. *(Lesson 11-5)*

66. $(12, 20), (16, 34)$
67. $(-18, 7), (2, 15)$
68. $(-2, 5), \left(-\frac{1}{2}, 3\right)$

Solve each equation by using the Quadratic Formula. Approximate irrational roots to the nearest hundredth. *(Lesson 10-4)*

69. $m^2 + 4m + 2 = 0$
70. $2s^2 + s - 15 = 0$
71. $2n^2 - n = 4$

Getting Ready for the Next Lesson

PREREQUISITE SKILL Find each sum or difference.
*(To review **fractions**, see pages 798 and 799.)*

72. $\frac{8}{52} + \frac{4}{52}$
73. $\frac{7}{32} + \frac{5}{8}$
74. $\frac{5}{15} + \frac{6}{15} - \frac{2}{15}$
75. $\frac{15}{24} + \frac{11}{24} - \frac{3}{4}$
76. $\frac{2}{3} + \frac{15}{36} - \frac{1}{4}$
77. $\frac{16}{25} + \frac{3}{10} - \frac{1}{4}$

Practice Quiz 1

Lessons 14-1 and 14-2

Find the number of outcomes for each event. *(Lesson 14-1)*

1. A die is rolled and two coins are tossed.
2. A certain model of mountain bike comes in 5 sizes, 4 colors, with regular or off-road tires, and with a choice of 1 of 5 accessories.

Find each value. *(Lesson 14-2)*

3. ${}_{13}C_8$
4. ${}_{9}P_6$

5. A flower bouquet has 5 carnations, 6 roses, and 3 lilies. If four flowers are selected at random, what is the probability of selecting two roses and two lilies? *(Lesson 14-2)*